XIV. Critical Phenomena: Ferromagnetism

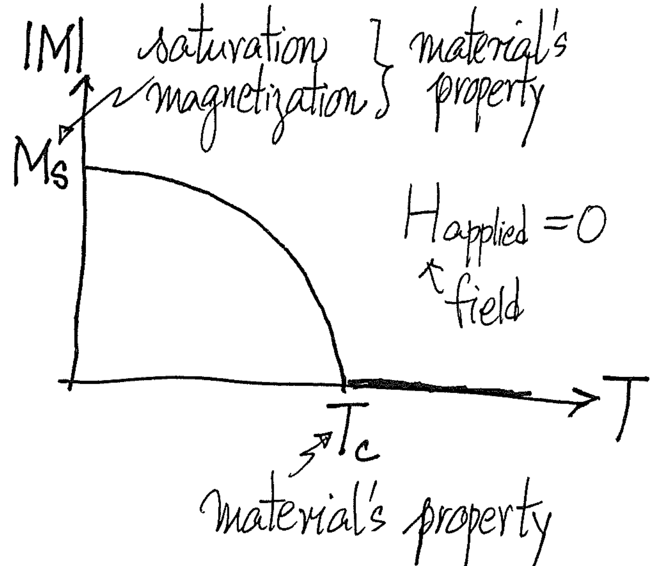
A. Key Features in Ferromagnetism

Spontaneous Magnetization (自發)

- No external applied magnetic field

$$\begin{cases} \underbrace{M \neq 0}_{\text{"spontaneous"}} \text{ for } T < T_c \leftarrow \text{Curie temperature} \\ M = 0 \text{ for } T > T_c \end{cases}$$

"spontaneous" ($\because$ no applied field to magnetize sample)



$|M| \sim (T_c - T)^{\beta}$

Critical point:

$(T = T_c, \ H_{applied} = 0)$

Material	M_s (10^6 A/m)	T_c (K)
Iron	1.75	1043
Cobalt	1.45	1404
Nickel	0.512	631
Gadolinium	2.00	289
Terbium	1.44	230
Dysprosium	2.01	85
Holmium	2.55	20

Sources: American Institute of Physics Handbook (D. W. Gray, Ed.) (New York: McGraw-Hill, 1963).

- "Universal behavior" when properties of different Ferromagnetic (FM) materials are viewed in <u>reduced quantities</u>

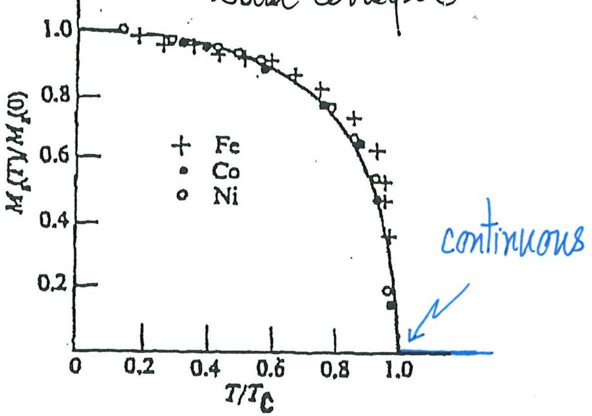
FM material A: Measure $M(T)$
$\Rightarrow M_s^{(A)}, T_c^{(A)}$

FM material B: Measure $M(T)$
$\Rightarrow M_s^{(B)}, T_c^{(B)}$

Look at $\underbrace{\dfrac{M(T)}{M_s} \text{ vs } \dfrac{T}{T_c}}$

Almost the same curve for different materials

Data Collapse



- For $T \lesssim T_c$, $M \sim (T_c - T)^\beta$ and <u>same</u> β for many different materials [c.f. vapor-liquid case]$^+$
  - M is the <u>order parameter</u> of FM transition
  - M changes <u>continuously</u> at $T_c \Rightarrow$ "Continuous phase transition" (transitions that are <u>NOT</u> first order)

    $T < T_c$, $M \neq 0$ Ferromagnetic phase (more ordered)

    $T > T_c$, $M = 0$ Paramagnetic phase (disordered)

$^+$ Recall the law of corresponding state. Here is another example.

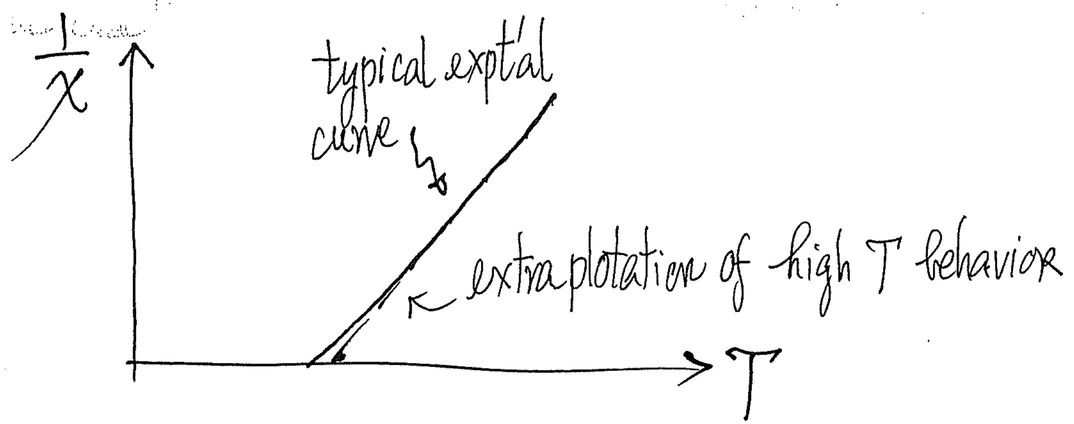
Paramagnetic behavior at $T > T_c$

$T > T_c$, $H_{applied} = 0$, $M = 0$ (no spontaneous magnetization)

With an applied field, $M \neq 0$ when $H_{applied} \neq 0$

Response is described by $M = \chi H_{applied}$

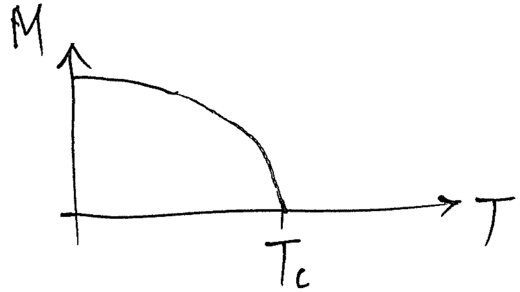
χ follows $\chi = \dfrac{C}{T - T_c}$ similar to Curie's law [recall $\chi \sim \frac{1}{T}$ for paramagnetic behavior]



It says, χ gets bigger and bigger as $T \to T_c$ from above.

As $T \to T_c$, χ diverges. OK! $\underset{\neq 0}{M} = \chi \underset{= 0}{H_{applied}}$ $\Rightarrow$ χ diverges when sample becomes FM!

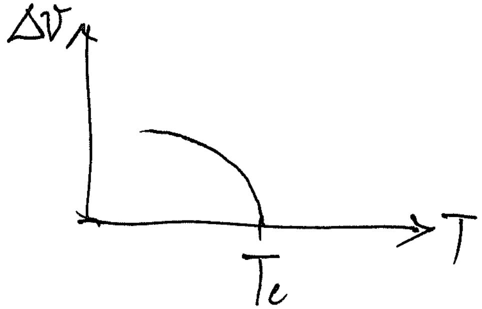
Ferromagnetic-Paramagnetic Transition



Ordered $\leftrightarrow$ Disordered

$M \sim (T_c - T)^{\beta}$

Liquid-vapor Transition



Ordered - Disordered

$\Delta v \sim (T_c - T)^{\beta'}$

It turns out that very different physical scenarios may carry the <u>same value</u> of <u>critical exponent</u>!

$\boxed{\text{This is what physicists meant by universal behavior.}}$

[Magnetic system and liquid-vapor system behave the same way near the critical point]

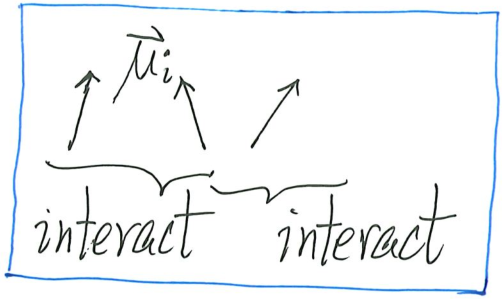
B. Hints from theory of Paramagnetism

- Atoms/ions → tiny magnet moments ↗

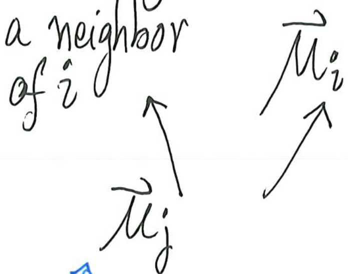
  they don't interact and only respond to an external magnetic field
  ⇒ Paramagnetism $M = M_S \tanh\left(\frac{\mu_B B_{app}}{kT}\right)$
  for $\mu = \pm\mu_B$ (two-level)

  ↙ Ferromagnetism: $B_{app} = 0$, $M \neq 0$

- Magnetic Moment $\vec{\mu}_i$ (i^{th} atom/ion) experiences an internal and local magnetic field due to neighboring magnetic moments



Ferromagnetic Interaction



$\therefore$ $\boxed{\underbrace{-\tilde{J}_{ij}\,\vec{\mu}_i \cdot \vec{\mu}_j}_{\text{an energy}}}$ ⟶ if $\vec{\mu}_i \parallel \vec{\mu}_j$, energy is lowest

["magnetic moment i, you had $\underbrace{\text{better}}_{\text{meaning: lower energy}}$ align your $\vec{\mu}_i$ with my $\vec{\mu}_j$"]

1-page review on Paramagnetism

$J = \frac{1}{2}$ $(s = \frac{1}{2})$ — $+\mu_B B$ (anti-align with applied B) — $-\mu_B B$ (align with applied B) (two-level system)

$$Z = z^N, \quad z = e^{\beta \mu_B B} + e^{-\beta \mu_B B} = 2\cosh\left(\frac{\mu_B B}{kT}\right)$$

$$\langle \mu_z \rangle = \mu_B \tanh\left(\frac{\mu_B B}{kT}\right)$$ (will use this result extensively)

one moment

<u>Whole system:</u> $N\langle \mu_z \rangle = N\mu_B \tanh\left(\frac{\mu_B B}{kT}\right)$

Per unit volume $M = \frac{N}{V}\langle \mu_z \rangle = \frac{N}{V}\mu_B \tanh\left(\frac{\mu_B B}{kT}\right) = M_s \tanh\left(\frac{\mu_B B}{kT}\right)$

Magnetization

saturation magnetization

B-field acting on each independent (non-interacting) magnetic moment

The underlying Hamiltonian is:

$$H_{para} = \sum_i -\vec{\mu}_i \cdot \vec{B}_{applied} \quad \text{with } \mu_{iz} = \begin{cases} +\mu_B & \text{(aligned with } \vec{B}_{applied}) \\ -\mu_B & \text{(anti-aligned with } \vec{B}_{applied}) \end{cases}$$

(cover all magnetic moments)

Alternative forms of H_{para} in magnetism

- $$H_{para} = \sum_{i \text{ (all moments)}} -\vec{\mu}_i \cdot \vec{B}_{applied}$$

When $\mu_{iz} = \begin{cases} +\mu_B & (\text{aligned with } \vec{B}_{applied} = B_{applied}\hat{z}) \\ -\mu_B & (\text{anti-aligned with } \vec{B}_{applied}) \end{cases}$

$$-\vec{\mu}_i \cdot \vec{B}_{applied} = \begin{cases} -\mu_B B_{applied} \\ +\mu_B B_{applied} \end{cases} \text{ two values}$$

$$\therefore H_{para} = (-\mu_B B_{applied}) \sum_i \sigma_i$$

$$= \underbrace{(-\mu_B B_{applied})}_{\text{an energy indicating how strong } B_{applied} \text{ is}} \sum_i S_i$$

$\sigma_i = \begin{cases} +1 & (\text{aligned with } B_{applied}) \\ -1 & (\text{anti-aligned with } B_{applied}) \end{cases}$ (related to σ_z's eigenvalues) Pauli matrix

$S_i = \begin{cases} +1 & (\text{aligned with } B_{applied}) \\ -1 & (\text{anti-aligned with } B_{applied}) \end{cases}$

In statistical mechanics books / Ising model literature, the term representing the effect is an external magnetic field is often written as

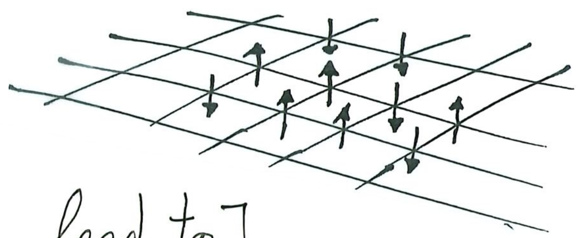
$$H_{para} = -B \sum_i S_i \quad \text{OR} \quad -H \sum_i S_i \quad \text{OR} \quad -B \sum_i \sigma_i \quad \text{OR} \quad -H \sum_i \sigma_i$$

this is an energy ($\mu_B B_{applied}$)

an energy (same as B)

C. Model : The Ising Model as the simplest model

- Retain the minimum (essential) physics that is key to understand ferromagnetic to paramagnetic transitions
- $\vec{\mu}_i$: Assume $\mu_{z,i}$ taking on only two values $(+\mu_B, -\mu_B)$ "Spin-½"
  [in general, can take on several $\mu_{z,i}$ values (quantum number J)]
- $\vec{\mu}_i$ interacts with <u>nearest neighboring</u> $\vec{\mu}_j$'s <u>only</u>
  (e.g. 2D square lattice, $\vec{\mu}_i$ only interacts with its 4 ($z=4$) nearest neighbors)



2D square lattice

[A fascinating idea: short-range (n.n.) interaction can lead to long-range order (ferromagnetism)]

	$\uparrow\ \uparrow$	and	$\downarrow\ \downarrow$	$\Rightarrow \varepsilon_{lower}$	
S_i :	$+1\ \ +1$		$-1\ \ -1$		
	$\uparrow\ \downarrow$	and	$\downarrow\ \uparrow$	$\Rightarrow \varepsilon_{higher}$	
S_i :	$+1\ \ -1$		$-1\ \ +1$		

$$\varepsilon_{higher} - \varepsilon_{lower} = \underbrace{2J}_{\text{want it to be "2J"}}$$

an energy

$T = 0$, then $\vec{\mu}_i$'s will align $\Rightarrow M = M_s$ (without $\vec{B}_{applied}$!)

$T \neq 0$, kT tends to randomize magnetic moments ($\because -TS$ entropy effect)
but $-\tilde{J}$ induces magnetic moments to align
$\Rightarrow$ standard statistical mechanics situation

Perhaps for some T_c, $T > T_c$, kT wins $\Rightarrow$ zero net alignment ($M=0$)
$T < T_c$, $-\tilde{J}$ wins $\Rightarrow$ net alignment ($M \neq 0$ even $B_{app}=0$)
$\therefore$ We saw a physical path through the phenomenon!

Physics is an experimental science

T_c (expt) or kT_c informs us about the magnitude of interaction energy

$\underbrace{T_c(\text{iron}) = 1043\,K, \; T_c(\text{nickel}) = 631\,K}_{kT_c \sim 0.05 - 0.1\,eV}$ (that's why we have magnets at room temperature)

(strong compared with EM prediction of how $\vec{\mu}_i, \vec{\mu}_j$ interact)
(the interaction has QM origin)

Write interaction energy as

$$\boxed{-J_{ij}\, S_i S_j}$$ with $\begin{cases} S_i \text{ taking on } +1 \text{ or } -1 \\ S_j \text{ taking on } +1 \text{ or } -1 \end{cases}$

$S_i \; S_j$

aligned: $(+1,+1)$ and $(-1,-1)$ give $S_i S_j = 1$ and energy $-J_{ij}$

anti-aligned: $(+1,-1)$ and $(-1,+1)$ give $S_i S_j = -1$ and energy $+J_{ij}$

differ by $2J_{ij}$

- J_{ij} is an energy characterizing the strength of interaction
- $J_{ij} > 0$, S_i and S_j tend to align $\Rightarrow$ FM interaction

[$J_{ij} < 0$, S_i and S_j tend to anti-align $\Rightarrow$ Anti-ferromagnetic interaction]

- Assuming $J_{ij} = J$ (same) for all nearest-neighbors (ij), the interaction energy (Hamiltonian) is

$$\boxed{E(\{S_i\}) = -J \underbrace{\sum_{\langle ij \rangle}}_{\text{sum over all distinct nearest-neighbor pairs}} S_i S_j}$$

Ising model with no external applied magnetic field

When there is an applied (external) $\vec{B}_{applied}$, then

add in a term

$$E_B(\{S_i\}) = -\underset{\substack{\uparrow \\ \text{an energy } (\mu_B B_{applied}) \\ \text{representing the strength} \\ \text{of the applied field}}}{B} \sum_i S_i \qquad S_i = +1 \text{ or } -1$$

$\therefore$ J (internal field), B (applied external field)

J: ↑ an energy

B: ↑ an energy

Ising Model

$$\boxed{E(\{S_i\}) = -J\sum_{\langle ij\rangle} S_i S_j - B\sum_i S_i} \qquad (1)$$

Ising Model Hamiltonian

interaction between magnetic moments (or people called them spins)

energy representating external applied field

- this competes with kT
- S_i takes on $+1$ or -1
- When $B=0$, can spins align even at finite temperature?
- Can study Ising model on 1D chain (exactly solvable), 2D lattices (some exactly solvable), 3D lattices (no exact solution), 4D lattices, ...
- Lenz (1920) constructed the model for Ising (1925) to study in his thesis

Remarks

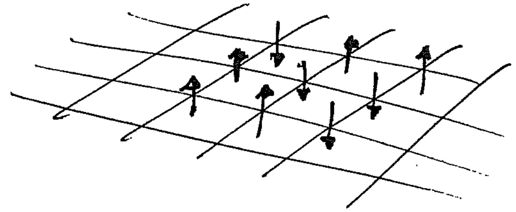
- $\langle S_i \rangle$ is a number between $+1$ and -1
- $\mu_B \langle S_i \rangle = \langle \mu_{i,z} \rangle = \langle \mu_z \rangle$ (if same for all locations i)

  is the average magnetic dipole moment (per dipole moment)
- Want to see if $\langle S_i \rangle \neq 0$ (thus $\langle \mu_z \rangle \neq 0$) in Ising Model even when $\underbrace{B=0}_{\text{no applied field}}$
- Turn off interaction $(J=0)$, $-B\sum_i S_i = H_{para}$ is the paramagnetism problem

  that can be exactly solved $\left(\langle \mu_z \rangle = \mu_B \tanh\left(\frac{\mu_B B}{kT}\right) \right)$

  OR $\langle S \rangle = \tanh\left(\frac{B}{kT}\right)$

D. Ising Model: What can be done, formally?



$$E(\{S_i\}) = -J\sum_{\langle ij\rangle} S_i S_j - B\sum_i S_i \quad (1)$$

Consider: N moments ("spins") on a 2D square lattice

In Stat. Mech., want to evaluate $Z = \sum_{\text{all } N\text{-spin states } \{S_i\}} e^{-\frac{E(\{S_i\})}{kT}}$

- What to sum over?

  2^N strings of the form $\{S_1, S_2, \ldots, S_N\}$ with $S_i = \begin{cases} +1 \\ -1 \end{cases}$

- What goes into $e^{-\frac{E(\{S_i\})}{kT}}$?

  - For every string $\{S_i\}$, calculate $E(\{S_i\})$ from Eq.( ) and evaluate one term in Z
  - Repeat for each of 2^N strings (2^N is a huge number for, say, $N=100^2$)
  - Done! In principle, after getting $Z(T,V,N)$

Can this be done?

- Analytically?
  - → 1D chain: Yes ($T_c = 0$, no FM state at any $T \neq 0$)
  - → 2D square lattice: Yes (but not so easy[+]), other lattices: No!
  - → 3D simple cubic or other lattice: No!
- Numerically?
  - → Any dimension? (Write a program to evaluate $Z(T, N, B)$ exactly?) [huge # of $\{S_i\}$'s]
  - → Any dimension? (An algorithm to carry out the importance sampling implied by the canonical ensemble) [Monte-Carlo simulation]
- Approximately?
  - $\boxed{\text{Mean field theories}}$
  - $1+\varepsilon$ ($\varepsilon \ll 1$) dimension; $4-\varepsilon$ ($\varepsilon \ll 1$) dimension
  - Renormalization methods

[+] Done by Onsager, C.N. Yang

E. Simplest Mean-Field Theory by physical arguments

- Don't want (know how) to handle interaction terms $-J\sum_{\langle ij\rangle} S_i S_j$
- But know how to handle single moment interacting with a field $-B\sum_i S_i$

Idea: Approximate internal field due to interactions as a mean field

$\Rightarrow$ "reduce"+ two-moment terms to single-moment in mean-field terms

for S_i: $$-J\sum_{j\in\{nn\text{ of }i\}} S_i S_j \approx -J\sum_{j\in\{n.n.\text{ of }i\}} S_i \langle S_j\rangle$$

$\langle S_j\rangle$: a mean number (same for all $j \in \{nn \text{ of } i\}$)

$$= -\underbrace{J(\langle S\rangle z)}\, S_i$$

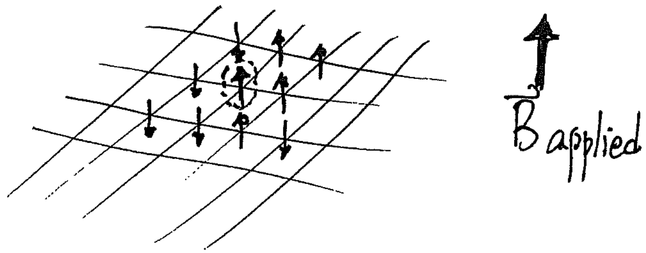
z: # n.n. (coordination number)

like an effective field (effective internal field) on S_i

$\approx$ an effective paramagnetic problem (if we know $\langle S\rangle$)

+ Usually called decoupling

Physical Picture



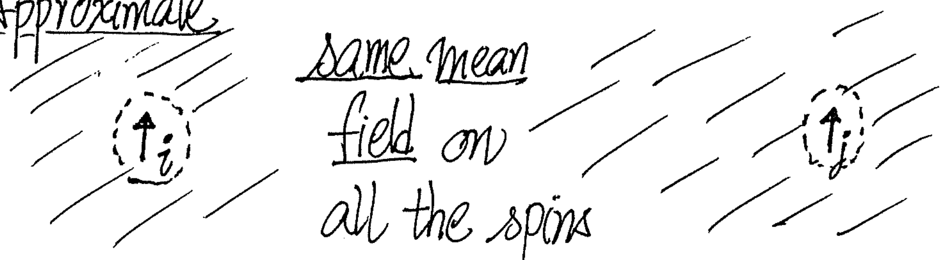
$\uparrow$ "feels" a field due to
(interaction with neighboring spins + $\vec{B}_{\text{applied}}$)

(i) Approximate as

some mean field due to surrounding spins

$\uparrow$ is under the influence of $\vec{B}_{\text{local}} = \underbrace{\vec{B}_{\text{mean field}}}_{\boxed{\text{yet-to-be-determined}}} + \vec{B}_{\text{applied}}$

(ii) Approximate



Argument: $\uparrow$ is nothing special

Deeper: Ignore differences in actual local fields at different spins
"ignore fluctuations"

[better approximation when there are more nearest neighbors]

(iii) Self-Consistency Condition

- Mean-field $= J \underset{\text{\# nearest neighbors}}{z} \langle S \rangle \propto \underbrace{\langle S \rangle}_{\text{to-be-determined (the unknown)}} \Rightarrow$ better alignment gives a strong mean field
- But a strong mean field is needed to get at better alignment

Alignment as quantified by $\langle S \rangle$ gives mean field and thus B_{local}
Mean field (thus B_{local}) gives $\langle S \rangle$ for quantifying Alignment

[†]

$\therefore$ $\langle S \rangle$ must be determined self-consistently !

† For those who know some atomic physics, the same type of argument is used in constructing the Hartree approximation for solving the single-electron atomic orbitals in multi-electron atoms.

$$E(\{S_i\}) = -J\sum_{\langle ij\rangle} S_i S_j - B\sum_i S_i$$ [interaction S_iS_j term hard to treat]

$$\approx -J\sum_{\langle ij\rangle} S_i \underbrace{\langle S_j\rangle} - B\sum_i S_i$$ [approximation]

stat. mech. average of neighboring spin gives <u>mean field</u> $\langle S_j\rangle = \langle S\rangle$

$$\approx -J\langle S\rangle z\sum_i S_i - B\sum_i S_i$$ [become problem of <u>independent spins</u> in a field, as in paramagnetism]

$\langle S\rangle$: to-be-determined

z: # nearest neighbors

["$\dfrac{J\langle s\rangle z}{\mu_B}$" is the strength of mean field (mean internal field)]

like an effective paramagnetism problem

$$\boxed{E(\{S_i\}) \approx -\underbrace{(J\langle s\rangle z + B)}_{\text{"a field acting on spin } i\text{"}}\sum_i S_i \qquad (2)}$$

$J\langle s\rangle z$: Mean-field (an energy)

B: external applied field (if present) (an energy)

z = coordination number of lattice

(e.g. $z=4$ for square lattice)

$z=6$ for simple cubic lattice

Turning Paramagnetic result into a theory of Ferromagnetism

$$E(\{S_i\}) \cong -(J\langle S\rangle z + B)\sum_i S_i \quad (2), \qquad S_i = +1, -1$$

(c.f. $H_{para} = -B\sum_i S_i$ for paramagnetism)

So, $(J\langle S\rangle z + B)$ acts like an effective field (actually an energy) on individual moments, BUT $\langle S\rangle$ is not known

Recall: Paramagnetism

$$\langle \mu_z\rangle = \mu_B \tanh\left(\frac{\mu_B B}{kT}\right) \quad \text{for } H_{para} = -\mu_B B\sum_i S_i$$

thus $$\langle S\rangle = \tanh\left(\frac{B}{kT}\right) \quad \text{for } H_{para} = -\underbrace{B}_{\text{energy}}\sum_i S_i$$

For Ising Model,

$$\boxed{\langle S\rangle = \tanh\left[\frac{J\langle S\rangle z}{kT} + \frac{B}{kT}\right]} \quad (3)$$

a self-consistent equation to solve for $\langle S\rangle(T)$

Usually, call $\langle S \rangle = m$ ($\langle S \rangle \mu_B$ is $\langle \mu \rangle$, m is a number between $+1$ and -1)

average magnetic dipole moment (in μ_B) per atom/ion

$$\boxed{m = \tanh\left[\frac{Jz\,m}{kT} + \frac{B}{kT}\right]} \qquad (4)$$

This is the "mean-field" (theory) equation to solve for $m(T,B)$

Question: Look for possible spontaneous magnetization at low temperature

$\underbrace{\text{spontaneous}}_{B = \mu_B B_{applied} = 0 \text{ (no applied field)}}$

$$\text{Study} \qquad m = \tanh\left(\frac{Jz}{kT}\,m\right) \qquad (5) \qquad \text{self-consistent equation for } m(T)$$

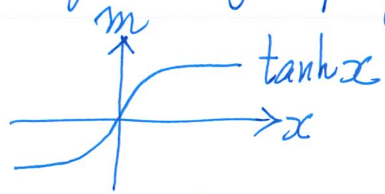
(i) Solve numerically

(ii) Think graphically $\quad \frac{Jz}{kT}\,m = x \;\Rightarrow\;$

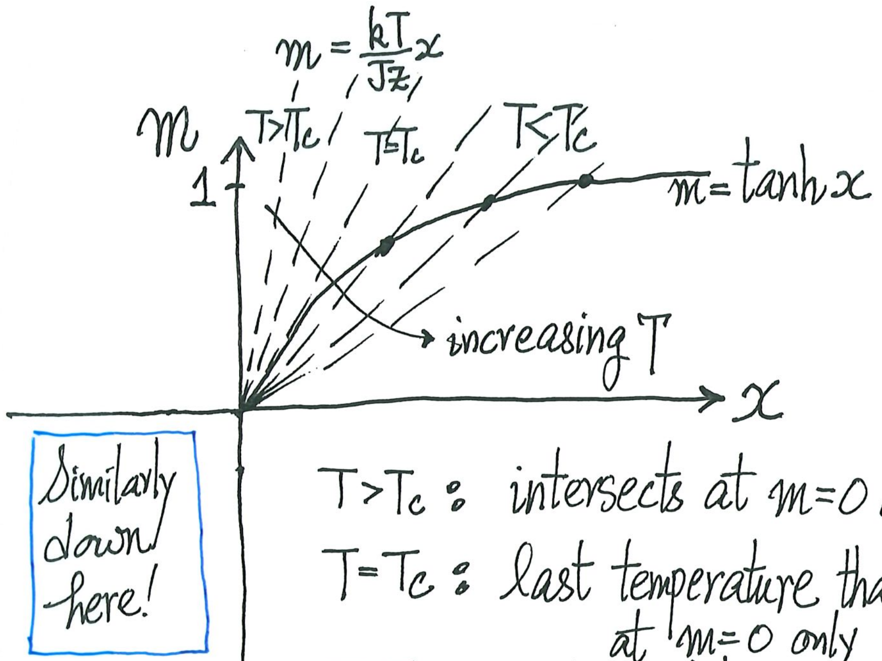
$$\boxed{\begin{array}{l} m = \dfrac{kT}{Jz}\,x \\ m = \tanh x \end{array}}$$

$\rightarrow$ (a straight line of slope $\frac{kT}{Jz}$)

$\rightarrow$



intersections of two curves obey $m = \tanh\left(\frac{Jz}{kT}\,m\right)$



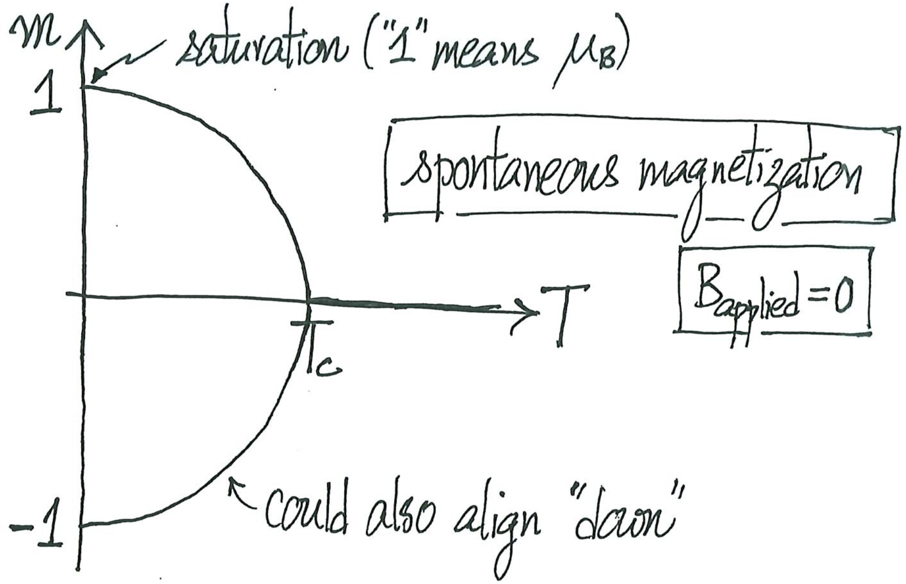
$T>T_c$: intersects at $m=0$ only

$T=T_c$: last temperature that intersects at $m=0$ only

$T<T_c$: $m \neq 0$ solutions arise

$T \to 0$: $m \to 1$ (saturation for one spin)

$\Rightarrow$

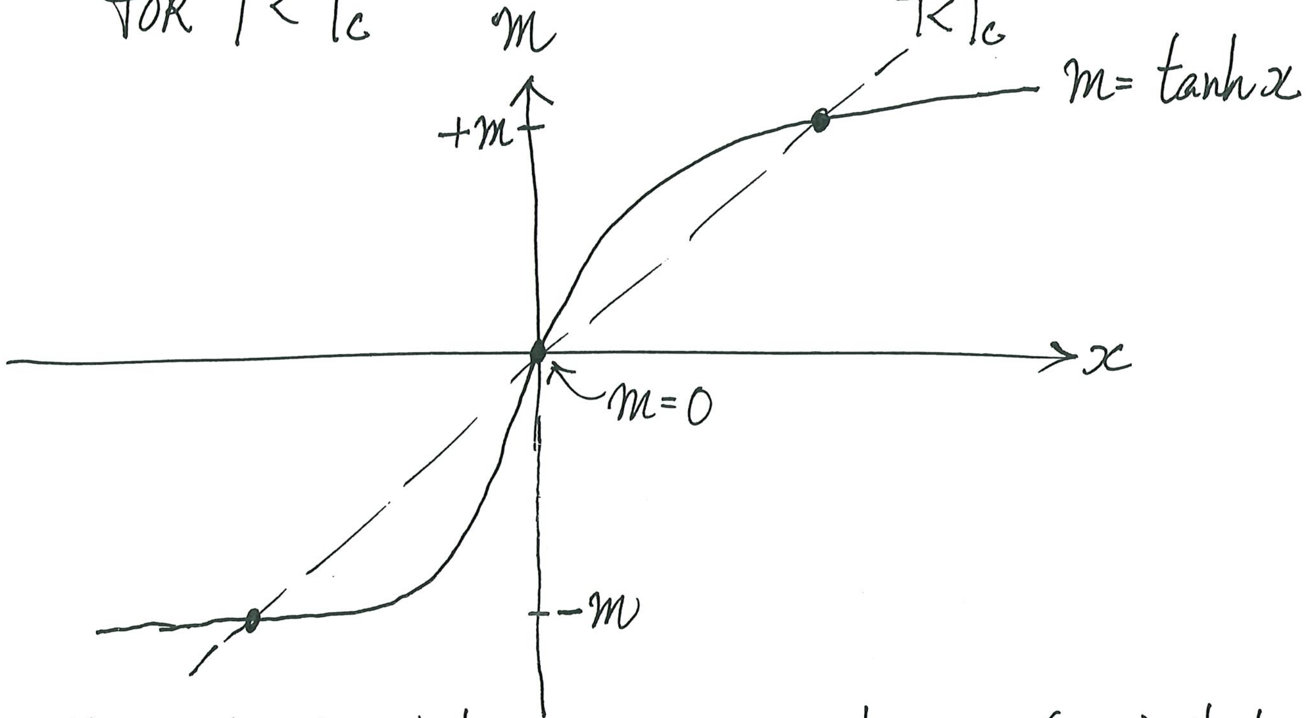


$m = \langle S_i \rangle$

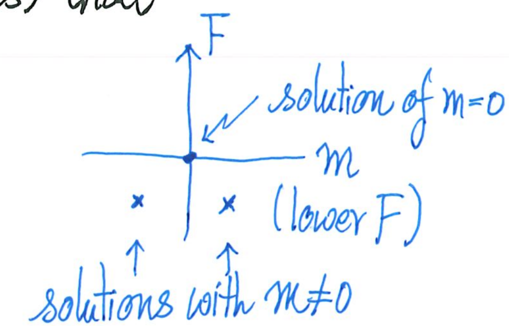
Formally,

$$m = \tanh\left(\frac{Jzm}{kT}\right)$$ allows 3 solutions

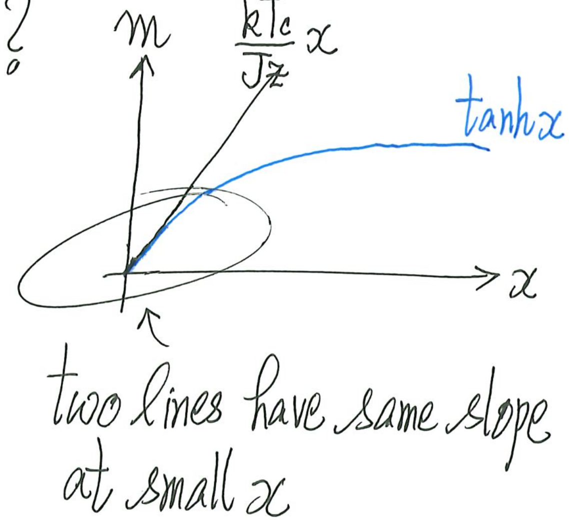
for $T < T_c$



The physically realized solution(s) is (are) the one (ones) that gives (give) a minimum Helmholtz free energy



What is the critical temperature within mean-field theory?



$$m = \tanh x \approx x = \frac{Jz}{kT_c} m \Rightarrow 1 = \frac{Jz}{kT_c}$$

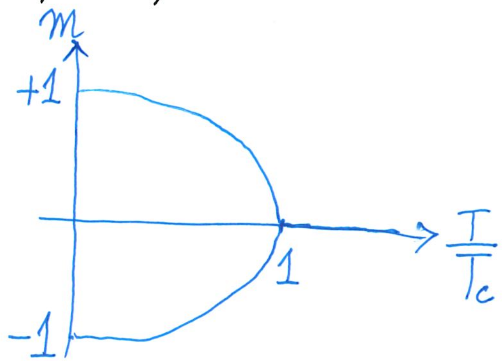
$$\Rightarrow \boxed{kT_c^{(MF)} = zJ}^{+} \quad \text{or} \quad \boxed{T_c^{MF} = \frac{zJ}{k}} \qquad (6)$$

- In terms of $T_c^{(MF)}$, the mean-field equation is $m = \tanh\left(\frac{T_c}{T} \cdot m\right)$
- m (+1 to −1) is also M/M_s ← saturation magnetization of a sample

∴ Mean-field theory suggests

$$\frac{M}{M_s} = \tanh\left(\frac{T_c}{T} \cdot \frac{M}{M_s}\right) \qquad (7)$$

T_c, M_s are substance-dependent



for collapsing experimental data (works quite well, see p. XIV-2)

+ kT_c really indicates the strength of interaction, as discussed.

Summary: Steps in setting up mean-field theory

- Decoupling the coupling term $S_i S_j \approx S_i \langle S_j \rangle$

  [Interacting system $\approx$ effective non-interacting system]

- Evaluating $\langle S \rangle$ using the approximated $E_{MF}(\{S_i\})$ to set up self-consistent equation(s)

- Same idea can be applied to many other problems

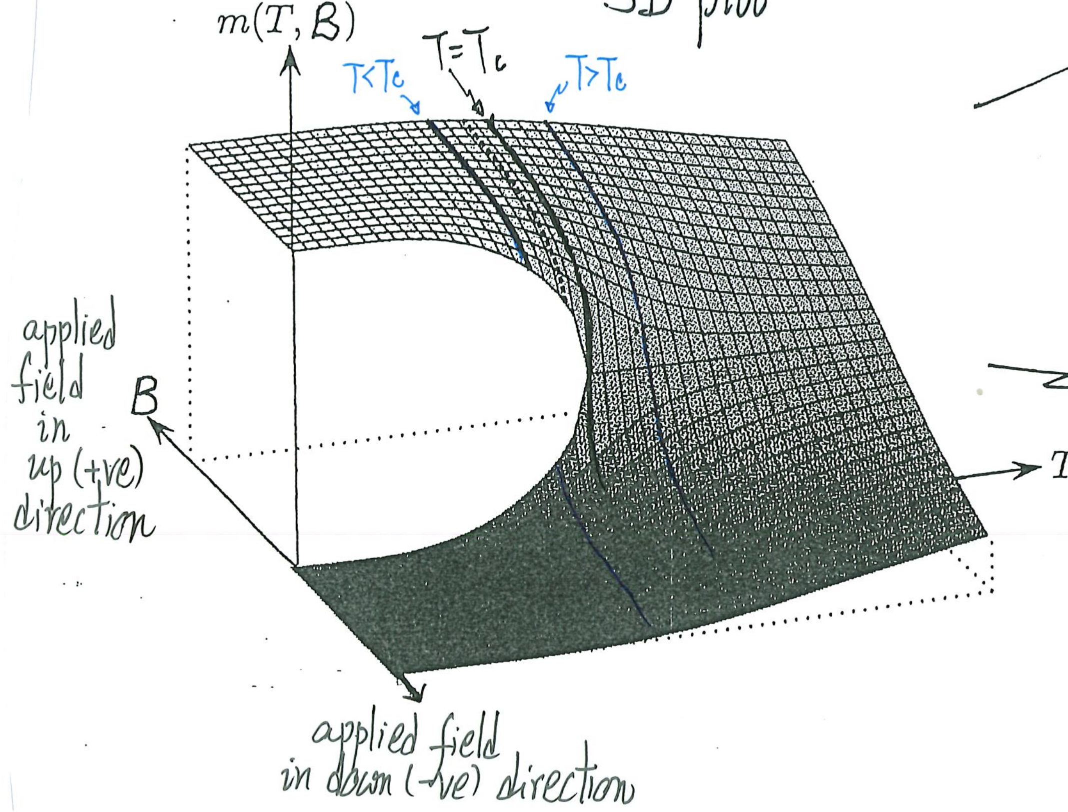
F. Critical Behavior as predicted by Mean field theory

MF equation $$m = \tanh\left[\frac{Jz}{kT} m + \frac{B}{kT}\right] \quad (4)$$

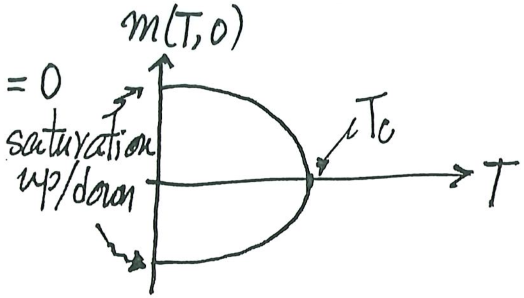
The critical point is $(T = T_c,\ B = 0)$
no applied field

Given T and B, solve for $m(T, B)$

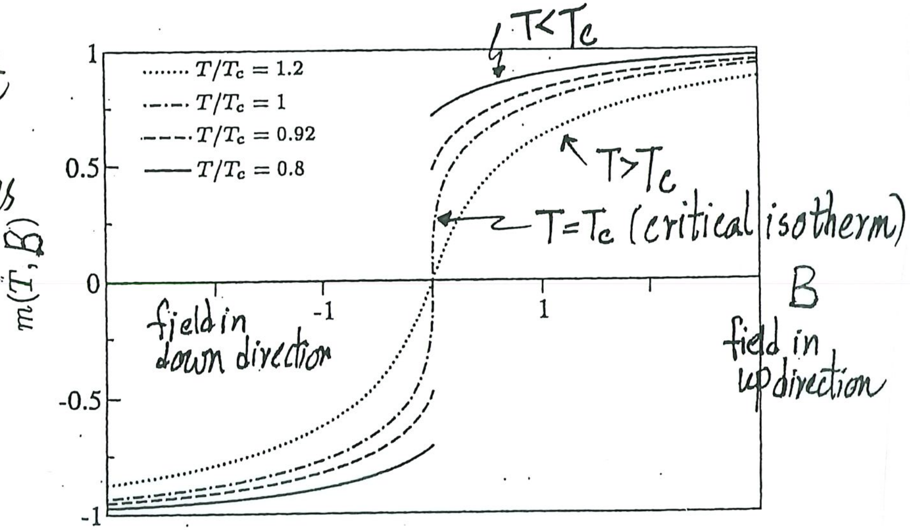
3D plot



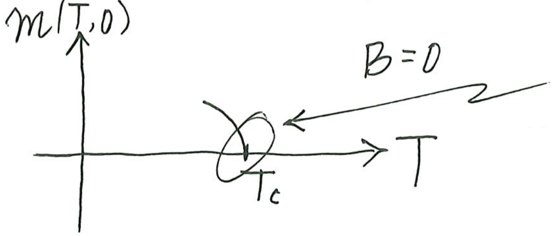
2D Cut at B = 0



2D cuts at constant temperatures



$(T = T_c, B = 0)$ is the critical point, m is the order parameter



$m \sim (T_c - T)^{1/2}$, $\beta_{MF} = 1/2$

$B = 0$, MF Eq.(4) becomes

$$m = \tanh\left(\frac{Jz}{kT} m\right) = \tanh\left(\underbrace{\frac{Jz}{kT_c}}_{=1 \text{ (determines } kT_c)} \cdot \frac{T_c}{T} m\right) = \tanh\left(\frac{T_c}{T} \cdot m\right)$$

Near critical point, $T \lesssim T_c$, $m \ll 1$, $\tanh x \approx x - \frac{x^3}{3}$ for $x \ll 1$

$$m \approx \frac{T_c}{T} m - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3$$

($m = 0$ solution is for $T > T_c$, irrelevant here)

$$\Rightarrow m^2 = 3\left(\frac{T}{T_c}\right)^3 \left[\frac{T_c}{T} - 1\right] = 3\left(\frac{T}{T_c}\right)^3 \left(\frac{T_c - T}{T}\right)$$

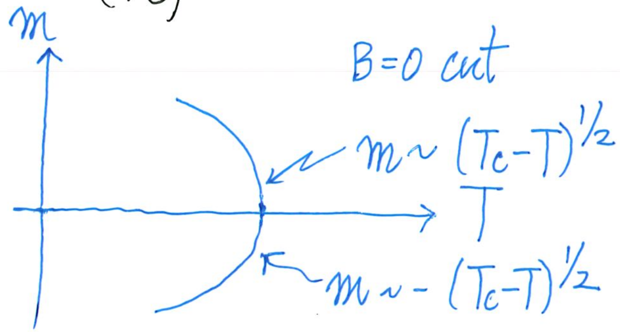
(after cancelling m ($m \neq 0$))

as $T \approx T_c$, $m^2 \approx 3 \frac{1}{T_c}(T_c - T)$ $\Rightarrow m = \pm\left(\frac{3}{T_c}\right)^{1/2} (T_c - T)^{1/2}$ (7)

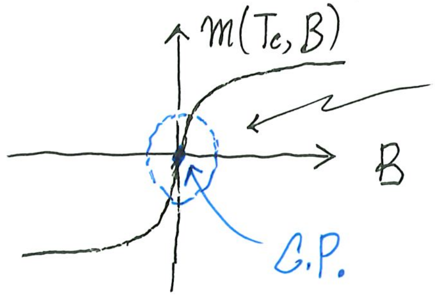
The critical exponent β is

$m \sim (T_c - T)^{\beta}$

$\therefore \beta_{MF} = 1/2$ (8)



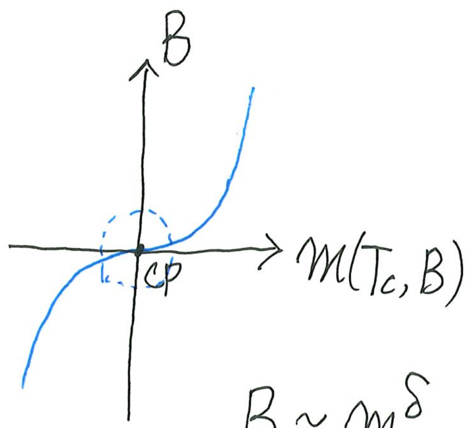
Look at $T=T_c$ cut, then $m(T_c,B)$ is



What is the behavior?

$$m(T_c,B) \sim |B|^{1/\delta} \qquad (9)$$

OR



δ is another critical exponent

$$B \sim m^{\delta} \qquad (10)$$

$T=T_c,\ B\neq 0$:
$$m = \tanh\left(\frac{Jz}{kT_c}m + \frac{B}{kT_c}\right) = \tanh\left(m+\frac{B}{kT_c}\right)$$

(see figures, we are interested in $|m| \ll 1$, $\frac{|B|}{kT_c} \ll 1$)

leading term

$$m \approx \left(m+\frac{B}{kT_c}\right) - \frac{1}{3}\left(m+\frac{B}{kT_c}\right)^3 \Rightarrow \frac{B}{kT_c} = \frac{1}{3}\left(m+\frac{B}{kT_c}\right)^3 = \frac{1}{3}\left(m^3 + \frac{3m^2B}{kT_c} + \frac{3mB^2}{(kT_c)^2} + \frac{B^3}{(kT_c)^3}\right)$$

[$T=T_c$, $m=0$ if $B=0$, $\therefore$ m due to B. $B\sim m^3$ is leading term, $m^2B \sim m^5$, $mB^2 \sim m^7$, $B^3 \sim m^9$]

$$\therefore \boxed{B \sim \frac{kT_c}{3}m^3 \sim m^3} \quad (11) \Rightarrow \delta_{MF} = 3 \qquad (12)$$

OR

$$m = \left(\frac{3}{kT_c}\right)^{1/3} B^{1/3} \Rightarrow \boxed{m \sim \mathrm{sign}(B)\,|B|^{1/3}} \qquad (13)$$

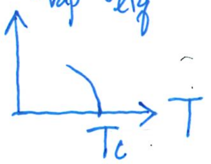
Van der Waals Equation of state

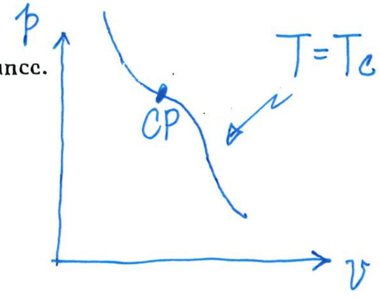
$\Delta v \sim (T_c - T)^{1/2}$; $\Delta p \sim -(\Delta v)^3$ for $T = T_c$

Mean Field Theory of Ferromagnetism

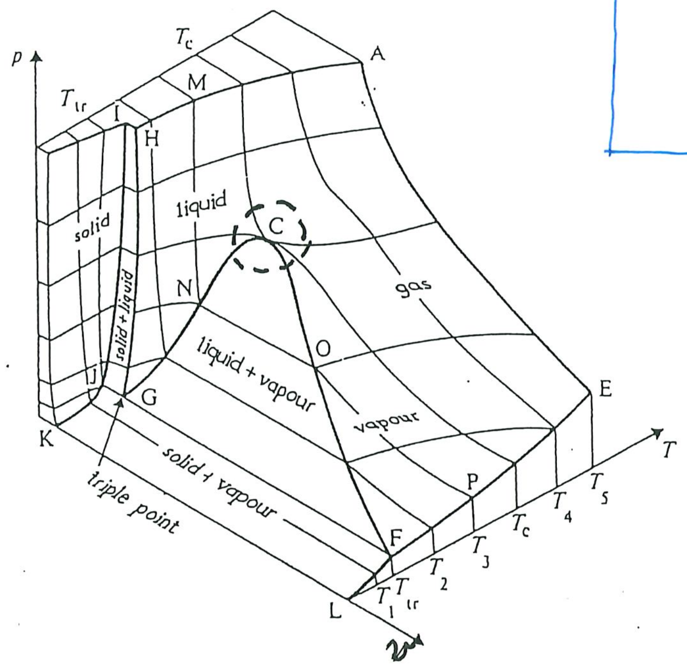
$m \sim (T_c - T)^{1/2}$; $B \sim m^3$ for $T = T_c$

$\Delta V = V_{vap} - V_{liq}$

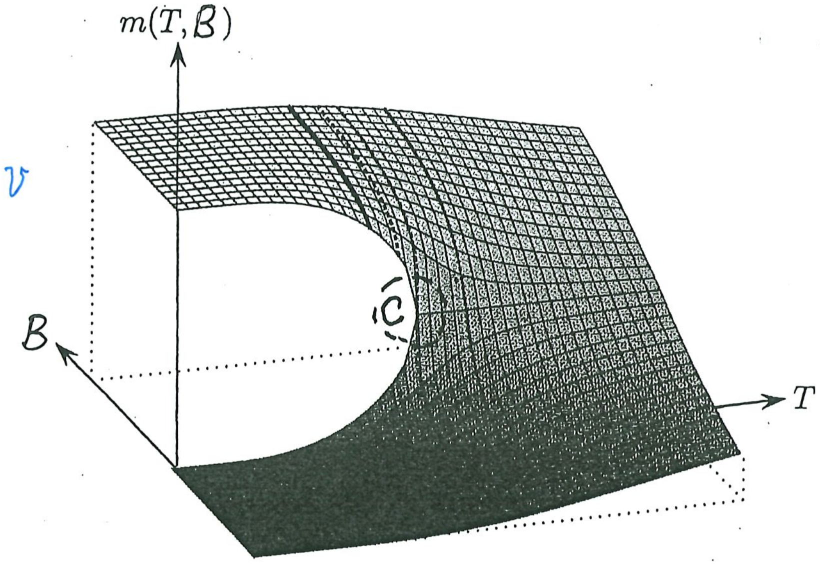




The p–V–T relation of a pure substance.



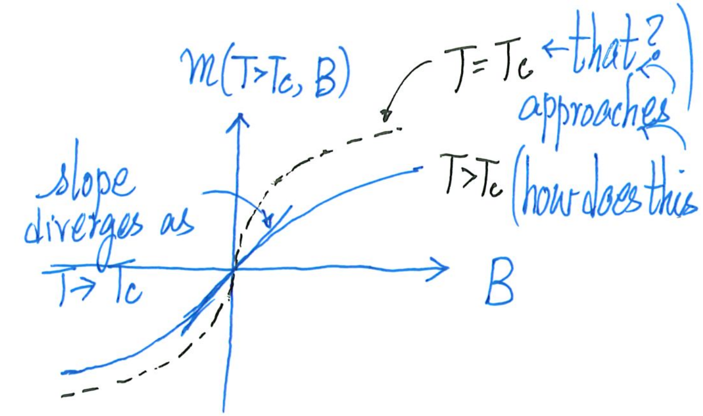
Ising Model



- Two seemingly different physical systems behave the same way near the critical point!
- Two seemingly different theories give same behavior near critical point! Universality! (普適)

How about $T > T_c$ paramagnetic behavior?

- $T > T_c$, no spontaneous magnetization
- With applied $B \neq 0$, there is $m \neq 0$
- Again start with $m = \tanh\left[\frac{T_c}{T} \cdot m + \frac{B}{kT}\right]$



Look for how m varies with B.

$$m \approx m\frac{T_c}{T} + \frac{B}{kT}$$

(consider $\frac{B}{kT} \ll 1$ (weak field) and see how m responds)

$$\Rightarrow m\left(1 - \frac{T_c}{T}\right) = \frac{B}{kT} \Rightarrow m\left(\frac{T - T_c}{T}\right) = \frac{B}{kT} \Rightarrow \underbrace{m = \frac{B}{k}\left(\frac{1}{T - T_c}\right)}_{\text{paramagnetic behavior}} \quad (14)$$

Recall: $\vec{M} = \chi \vec{H}$, with $\vec{M} \propto m$, $\vec{H} \propto B$

$$\therefore \chi \sim \frac{1}{T - T_c} \sim (T - T_c)^{-1} \quad \text{for } T \to T_c^+$$

agree with exp'tal observation $\sim (T - T_c)^{-\gamma}$ (11)

$$\therefore \boxed{\gamma_{MF} = +1} \quad (15)$$

Ising Model using Mean-field theory

$m \sim (T_c - T)^{\beta}$ $\quad \beta_{MF} = 1/2$

$B \sim m^3 \quad (T = T_c)\,, \quad \delta_{MF} = 3$

$\chi \sim \frac{1}{T - T_c} \quad (T > T_c)\,, \quad \gamma_{MF} = 1$

$\sim (T - T_c)^{-\gamma}$

these are called Mean-Field exponents

- usually deviate from accurate measured values and accurate numerical values by simulations
- but capture correct behavior near critical point, despite only qualitatively